Exercise 2

1. Flip a fair coin twice. What is the probability that you get two heads (HH)? What is the probability that you get heads followed by tails (HT)? Are these probabilities the same?

2. Flip a fair coin repeatedly until you get two heads in a row (HH). What is the probability that it takes n flips to win? (*suggestion*: go all the way up to $n = 8$ before making conclusions).

3. Flip a fair coin repeatedly until you get heads and tails in a row (HT). What is the probability that it takes n flips to win?

4. Based on the answers to point 2 and 3, is the probability to obtain a *large* value of n equal in the two cases? If it is not, which probability is the highest?

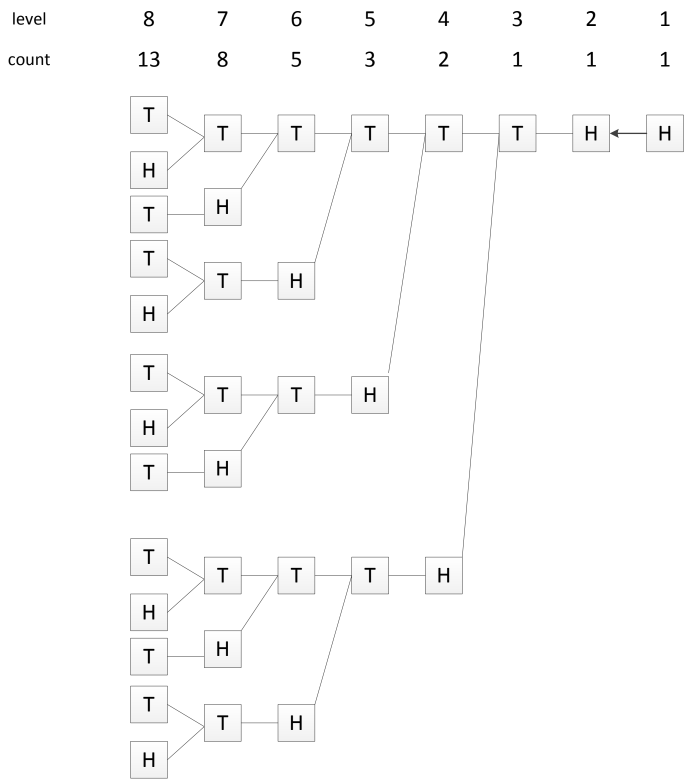
5. Let's play a game: we flip a coin repeatedly until either HH emerges (I win) or HT emerges (you win). Is the game fair (i.e., are the two players equally likely to win)?

Solution

1. The two probabilities are obviously equal, and each one is $(1/2)^2 = 1/4$

2. This is clearly a uniform probability model. The number of *n*-sequences is 2^n, whereas the number of *good* *n*-sequences can be computed by constructing a tree. The highest two levels of the tree are HH, and every time a level *j* is added, only feasible sequences are generated (i.e., you only add a child T to a parent H, whereas you add both children H and T to a parent T). The tree is in the figure below.
By counting the number of nodes at each level, one immediately gets that the number of *good* n-sequences is the $n-1$ number in the Fibonacci sequence S_{n-1}. Therefore:

$$P\{N_{HH} = n\} = \frac{S_{n-1}}{2^n}, \; n \geq 2$$



3. We repeat the same argument as before, and find that the number of *good* *n*-sequences is different. In fact, every node H spawns two children (H and T), whereas a node T spawns only a T. Therefore, the number of *good* *n*-sequences increases by one at each step, i.e., it grows linearly and is equal to *n*-1. Hence:

$$P\{N_{HT} = n\} = \frac{n-1}{2^n}, \; n \geq 2$$

4. Given the above, we can observe that $S_{n-1} > n-1$ starting from $n = 6$ (everyone knows that the Fibonacci sequence is superlinear). Therefore, the probability of observing long sequences is larger for HH.

5. The answer is counterintuitive. The game *is* fair, despite the fact that you observe longer sequences more often with HH. This can be proved by observing that the *only* *n*-sequence that can repeat infinitely without either player winning is {TT..TT}. As soon as *one* H appears, the winner is decided by the next flip. So, there is only *one* sequence that leads to the decisive flip, and it is {TTT….TT}H. Once we get at this point, both players have exactly 50% chance to win.