Exercise 2

A first aid unit admits patients of two types (critical and non-critical), whose daily number can be modeled through independent Poisson variables X and Y, whose parameters are λ and μ respectively.

1) Compute the mean and variance of the *overall* number of daily patients $Z = X + Y$.
2) Prove the above result formally.
3) Compute the probability that there are h critical patients, *and* that the total number of patients is m.
4) Compute the probability that there are h critical patients, *given that* that the total number of patients is m.
5) Identify the distribution obtained at point 4).
6) Assume that $\lambda = \mu = 20$. Compute the probability of having at least 60 patients of one type (whichever), given that you have 100 patients. Discuss the correctness of the approximation.

Un pronto soccorso ammette due tipi di pazienti (critici e non-critici), il cui numero giornaliero può essere modellato da variabili aleatorie di Poisson indipendenti X e Y, di parametro λ e μ rispettivamente.

1) Calcolare media e varianza del numero *totale* di pazienti $Z = X + Y$.
2) Dimostrare formalmente il risultato precedente.
3) Calcolare la probabilità che ci siano h pazienti critici *e* che il numero totale di pazienti sia m.
4) Calcolare la probabilità che ci siano h pazienti critici *dato che* il numero totale di pazienti è m.
5) Identificare il tipo di distribuzione ottenuta al punto 4).
6) Si assuma che $\lambda = \mu = 20$. Calcolare la probabilità di avere almeno 60 pazienti di un tipo (qualunque), dato che ci sono in tutto 100 pazienti. Discutere la correttezza delle approssimazioni.

Solution

1) The sum of independent Poisson variables is itself a Poisson variable. Its mean and variance are therefore $E[Z] = Var(Z) = \lambda + \mu$.
2) See the course handouts for a full proof.
3) The distribution is the following:

$$
\begin{aligned}
P\{X = h, Z = m\} &= P\{X = h, Y = m-h\} \\
&= P\{X = h\} \cdot P\{Y = m-h\} \quad \text{(by independence)} \\
&= \left(e^{-\lambda} \cdot \frac{\lambda^h}{h!}\right) \cdot \left(e^{-\mu} \cdot \frac{\mu^{m-h}}{(m-h)!}\right) \\
&= e^{-(\lambda+\mu)} \cdot \frac{\lambda^h \mu^{m-h}}{h!(m-h)!}
\end{aligned}
$$

4) Using the former result, we get:

$$
\begin{aligned}
P\{X = h \mid Z = m\} &= \frac{P\{X = h, Y = m-h\}}{P\{Z = m\}} \\
&= \frac{e^{-(\lambda+\mu)} \cdot \dfrac{\lambda^h \mu^{m-h}}{h!(m-h)!}}{e^{-(\lambda+\mu)} \cdot \dfrac{(\lambda+\mu)^m}{m!}} \\
&= \binom{m}{h} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^h \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{m-h}
\end{aligned}
$$

5) The above distribution is clearly a binomial, with $p = \dfrac{\lambda}{\lambda+\mu} < 1$.
6) The requested probability is:

$$
\begin{aligned}
&P\{\{X \geq 60 \mid Z = 100\} \cup \{X \leq 40 \mid Z = 100\}\} \\
&= 1 - P\{40 \leq X \leq 60 \mid Z = 100\}
\end{aligned}
$$

The above probability can be approximated using a Gaussian, as long as the following inequality holds:

$$
m \cdot p \cdot (1-p) > 10
$$

$$
\frac{\lambda \cdot \mu}{(\lambda+\mu)^2} > \frac{10}{m}
$$

Substituting the numbers in the text, we get $0.25 > 0.1$, which is true. Thus we have:

$$
\begin{aligned}
P\{40 \leq X \leq 60 \mid Z = 100\} &= P\left\{\frac{40.5 - m \cdot p}{\sqrt{m \cdot p \cdot (1-p)}} \leq \frac{X - m \cdot p}{\sqrt{m \cdot p \cdot (1-p)}} \leq \frac{59.5 - m \cdot p}{\sqrt{m \cdot p \cdot (1-p)}} \mid Z = 100\right\} \\
&= \Phi(1.9) - \Phi(-1.9) \\
&= -1 + 2 \cdot \Phi(1.9)
\end{aligned}
$$

The requested probability is thus $1 - [-1 + 2\Phi(1.9)] = 2(1 - \Phi(1.9)) \simeq 0.0574$.