Exercise 2

Consider the following function:

$$F(x) = \begin{cases} 0 & x \leq -5 \\ \dfrac{\alpha \cdot x + 5}{\beta + |x|} & x > -5 \end{cases}$$

Where α, β are *positive* constants.

1) Determine under what conditions $F(x)$ is a CDF.

Assume from now on that we are in the above conditions.

2) Compute the PDF of RV X, whose CDF is $F(x)$.
3) Determine under what further conditions $E[X]$ is finite.
4) Assuming $\beta = 5$, compute $E[X]$.

Esercizio 2

Si consideri la seguente funzione

$$F(x) = \begin{cases} 0 & x \leq -5 \\ \dfrac{\alpha \cdot x + 5}{\beta + |x|} & x > -5 \end{cases}$$

Con α, β costanti positive.

1) Determinare le condizioni sotto le quali $F(x)$ è una CDF.

Si assuma d'ora in avanti che le condizioni di cui sopra siano soddisfatte.

2) Calcolare la PDF della variabile aleatoria X, la cui CDF è $F(x)$
3) Determinare le condizioni ulteriori sotto cui $E[X]$ è finito
4) Assumendo $\beta = 5$, calcolare $E[X]$.

Solution

1) In order to be a CDF, the following should happen:
   a) $F(x)$ must be monotonic
   b) $\lim_{x\to-\infty} F(x)=0$
   c) $\lim_{x\to+\infty} F(x)=1$

b) always holds. c) holds if and only if $\alpha=1$.
As far as monotonicity is concerned, we observe that $F(x)$ is identically null for $x\le -5$, and that its derivative is:

$$F'(x)=\begin{cases}\dfrac{\beta+5}{(\beta-x)^2} & -5<x<0\\ \dfrac{\beta-5}{(\beta+x)^2} & x>0\end{cases}$$

when $x>-5$. Therefore, monotonicity is guaranteed if $\beta\ge 5$.
The conditions requested by 1) are $\alpha=1$, $\beta\ge 5$.

2) As per the computations above, it is:

$$f(x)=\begin{cases}0 & x<-5\\ \dfrac{\beta+5}{(\beta-x)^2} & -5<x<0\\ \dfrac{\beta-5}{(\beta+x)^2} & x>0\end{cases}$$

3) It is:

$$E[X]=\int_{-\infty}^{+\infty} x\cdot f(x)\,dx=\int_{-5}^{0} x\cdot\frac{\beta+5}{(\beta-x)^2}\,dx+\int_{0}^{+\infty} x\cdot\frac{\beta-5}{(\beta+x)^2}\,dx=$$

$$(\beta+5)\cdot\int_{-5}^{0}\frac{x}{(\beta-x)^2}\,dx+(\beta-5)\cdot\int_{0}^{+\infty}\frac{x}{(\beta+x)^2}\,dx$$

Now, the first integral is always finite (since its limits are), whereas the second may not be. After few algebraic passages, we obtain:

$$\int_{0}^{+\infty}\frac{x}{(\beta+x)^2}\,dx=\int_{0}^{+\infty}\frac{(\beta+x)-\beta}{(\beta+x)^2}\,dx=\left[\frac{\beta}{\beta+x}+\log|\beta+x|\right]_{0}^{+\infty}=+\infty\,.$$

Therefore, the expectation exists only if $\beta=5$.

4) Assuming $\beta=5$, the expectation is equal to:

$$\begin{aligned}E[X]&=(\beta+5)\cdot\int_{-5}^{0}\frac{x}{(\beta-x)^2}\,dx+(\beta-5)\cdot\int_{0}^{+\infty}\frac{x}{(\beta+x)^2}\,dx\\ &=10\cdot\int_{-5}^{0}\frac{x}{(5-x)^2}\,dx=10\cdot\left[\frac{-5}{x-5}+\log|x-5|\right]_{-5}^{0}\\ &=5-10\log(2)\end{aligned}$$