Exercise 2

A machine reads a string of random binary digits, which appear with probability p_0, p_1 respectively. The machine stops reading after it has read *at least one 0 and at least one 1*.
Assume that each binary digit is independent of the others, and call X_0, X_1 the RVs representing the number of 0s and 1s read *at the end* of an experiment.

1) Compute the PMFs of X_0, X_1. Check the normalization condition.
2) Compute $P\{X_0 > X_1\}$, and compute p_0 such that $P\{X_0 > X_1\} = 0.25$. Find a general relationship between $P\{X_0 > X_1\}$ and p_0.
3) Compute $E[X_0]$. Justify explicitly what happens in the limit cases $p_0 \to 0, p_0 \to 1$.
4) Are X_0, X_1 independent? Justify your answer

Una macchina legge una stringa di cifre binarie casuali, che appaiono con probabilità p_0, p_1 rispettivamente. La macchina smette di leggere dopo che ha letto *almeno uno zero ed almeno un uno*. Si assuma che ogni cifra è indipendente dalle altre, e si chiamino X_0, X_1 le variabili aleatorie che contano il numero di zeri ed uni letti *alla fine* di un esperimento.

1) Calcolare la PMF di X_0, X_1. Controllare le condizioni di normalizzazione
2) Calcolare $P\{X_0 > X_1\}$, e calcolare p_0 in modo tale che $P\{X_0 > X_1\} = 0.25$. Trovare una relazione generale tra $P\{X_0 > X_1\}$ e p_0.
3) Calcolare $E[X_0]$. Giustificare esplicitamente cosa accade nei casi limite $p_0 \to 0, p_0 \to 1$.
4) X_0, X_1 sono independenti? Giustificare la risposta.

Solution

1) This is a case of repeated trials, with $p_0 = 1 - p_1$. Let us start from X_0:

$$P\{X_0 = 1\} = \sum_{n=1}^{+\infty} P\{n \text{ consecutive 1s and one 0}\} + P\{0,1\}$$
$$= \sum_{n=1}^{+\infty} p_1^{\,n} \cdot p_0 + p_0 \cdot p_1$$
$$= p_0 \cdot \left(\frac{1}{1-p_1} - 1 + p_1 \right)$$
$$= 1 - p_0 + p_0 \cdot p_1$$
$$= p_1 \cdot (1 + p_0)$$

And, for $n > 1$, $P\{X_0 = n\} = P\{n \text{ consecutive 0s and one 1}\} = p_0^{\,n} \cdot p_1$

The normalization condition is:

$$P\{X_0 = 1\} + \sum_{n=2}^{+\infty} P\{X_0 = n\} = 1$$
$$\left[p_1 \cdot (1 + p_0) \right] + \left[\sum_{n=2}^{+\infty} p_0^{\,n} \cdot p_1 \right] =$$
$$\left[p_1 \cdot (1 + p_0) \right] + \left[p_1 \cdot \left(\frac{1}{1-p_0} - 1 - p_0 \right) \right] =$$
$$\left[p_1 \cdot (1 + p_0) \right] + \left[1 - p_1 \cdot (1 + p_0) \right]$$
$$= p_1 + p_1 \cdot p_0 + 1 - p_1 - p_1 \cdot p_0$$
$$= 1$$

Symmetrically, for X_1 we have $P\{X_1 = 1\} = p_0 \cdot (1 + p_1)$, $P\{X_1 = n\} = p_1^{\,n} \cdot p_0$, and the normalization condition holds as well.

2) The event $\{X_0 > X_1\}$ occurs when 1-terminated sequences of three or more digits are observed. Hence:

$$P\{X_0 > X_1\} =$$
$$\sum_{n=2}^{+\infty} P\{X_0 = n\} =$$
$$\sum_{n=2}^{+\infty} p_0^{\,n} \cdot p_1 =$$
$$1 - p_1 \cdot (1 + p_0)$$

Furthermore, the inequality that ensures that $P\{X_0 > X_1\} \geq 0.25$ is

$$1 - p_1 \cdot (1 + p_0) = 0.25$$
$$0.75 = (1 - p_0) \cdot (1 + p_0)$$
$$0.75 = 1 - p_0^{\,2}$$
$$p_0^{\,2} = 0.25$$
$$p_0 = 0.5$$

Setting $P\{X_0 > X_1\} = \pi$, we easily obtain from the above $P\{X_0 > X_1\} = \pi \Leftrightarrow p_0 = \sqrt{\pi}$.

3) From the formula, we obtain:

$$E[X_0] =$$
$$1\cdot P\{X_0 = 1\} + \left[\sum_{n=2}^{+\infty} n\cdot P\{X_0 = n\}\right] =$$
$$p_1\cdot(1+p_0) + p_1\cdot\left[\left(\sum_{n=2}^{+\infty} n\cdot p_0{}^n\right)\right] =$$
$$p_1\cdot(1+p_0) + p_1\cdot\left[\left(\sum_{n=1}^{+\infty} n\cdot p_0{}^n\right) - p_0\right] =$$
$$p_1\cdot(1+p_0) + p_1\cdot\left[\left(\sum_{n=1}^{+\infty} n\cdot p_0{}^n\right) - p_0\right] =$$
$$p_1\cdot(1+p_0) + p_1\cdot\left[\left(p_0\cdot\frac{1}{(1-p_0)^2} - p_0\right)\right] =$$
$$p_1\cdot(1+p_0) + \frac{p_0}{p_1} - p_0\cdot p_1 =$$
$$p_1 + \frac{p_0}{p_1}$$

We have $\lim_{p_0\to 1} E[X_0] = +\infty$. This can be explained by observing that, if no 1 ever appears, the sequence of 0s becomes infinitely long. Furthermore, we have $\lim_{p_0\to 0} E[X_0] = 1$. In fact, if no 0 ever appears, the only sequences that we may ever obtain are infinitely long sequences of 1s, *terminated by a* 0 (this is mandatory for the experiment to terminate). Hence the mean number of 0s will be equal to one.

4) The two RVs are *not* independent. In fact, $P\{X_0 = 2, X_1 = 2\} = 0$ if $a > 1, b > 1$. On the other hand, $P\{X_0 = 2\}\cdot P\{X_1 = 2\} \neq 0$.