EXERCISE 1: Consider a Markov Chain with an infinite state space $E = \{0, 1, 2, \ldots\}$ and transition probabilities:

$$p_{ij} = e^{-\lambda}\sum_{n=0}^{j}\binom{i}{n}p^n q^{i-n}\frac{\lambda^{j-n}}{(j-n)!}, \tag{3.19}$$

$$\text{where } p+q = 1,\ 0<p<1, \text{ and } \lambda>0 \tag{3.20}$$

Complete the following tasks:

1. describe the nature of the MC states;
2. write down the elements of the first row (p_{0j}) of the transition probability matrix, calculate their sum and give a justification of the value obtained;
3. generalize the previous result to a generic row *i*;
4. starting from

$$\pi_i = \sum_{j=0}^{\infty}\pi_j p_{ji} \tag{3.21}$$

prove that

$$\Pi(z) = e^{\lambda(z-1)}\Pi(1+p(z-1)) \tag{3.22}$$

where

$$\Pi(z) = \sum_{i=0}^{\infty}\pi_i z^i \tag{3.23}$$

is the z-transform of the steady state probabilities π_i;

5. by using the *Principle of Mathematical Induction* prove that

$$\Pi(z) = e^{\lambda(z-1)f_{n-1}(p)}\Pi(1+p^n(z-1)),\ n\ge 1 \tag{3.24}$$

where

$$f_{n-1}(p) = \sum_{k=0}^{n-1}p^k,\ n\ge 1 \tag{3.25}$$

6. by using (3.24) calculate the steady state probabilities π_i.

HINT. To develop point 6 above, let $n \to \infty$ in both members of (3.24). The result will be that of a z-transform of a very well known distribution.