Exercise 2

In a dice game, you roll two dice. If you obtain a 2, 3, or 12, you immediately lose. If, instead, you obtain a 7 or 11, you immediately win. If you roll a 4, 5, 6, 8, 9, or 10, that becomes your "objective". In this case, you keep rolling the dice until either the "objective" comes up again – in which case you win – or until a 7 comes up, in which case you lose.

1) Calculate the probability that you win/lose at the first roll
2) Calculate the probability that you obtain a 4 at the first roll, and you win/lose at the second roll
3) Generalize the previous result to the case when you win/lose at the *n*-th roll ($n>=2$)
4) Using the previous two results, compute the probability that you win *at all*
5) Assume that you can bet 100$ on you winning. Compute the mean value of your payoff

In un gioco di dadi il giocatore getta due dadi. Se ottiene un 2, 3 o 12, perde immediatamente. Se, invece, ottiene un 7 o un 11, vince immediatamente. Se, infine, ottiene 4, 5, 6, 8, 9, o 10, il valore ottenuto diventa l'*obiettivo*. In questo caso, il giocatore continua a lanciare dadi finché non ottiene nuovamente l'obiettivo - nel qual caso vince - o finché non ottiene 7, nel qual caso perde.

1) Calcolare la probabilità di vincere/perdere al primo lancio
2) Calcolare la probabilità di ottenere un 4 al primo lancio, e di vincere/perdere al secondo
3) Generalizzare il risultato precedente al caso di vincita/perdita all'*n*-simo lancio ($n>=2$)
4) Sulla base dei due risultati precedenti, calcolare la probabilità di vincita
5) Assumendo che si possono scommettere 100 euro, calcolare la media dell'ammontare di soldi che si ottengono.

Solution

1) Call Call $P_x = P\{x\}$

$$P\left\{\text{win } 1^{\text{st}}\right\} = P\{7,11\} = P_7 + P_{11} = \frac{6+2}{36} = \frac{8}{36};$$

$$P\left\{\text{lose } 1^{\text{st}}\right\} = P\{2,3,12\} = P_2 + P_3 + P_{12} = \frac{1+2+1}{36} = \frac{4}{36}$$

2) $P\left\{4\ 1^{\text{st}},\ \text{win } 2^{\text{nd}}\right\} = P\{4,4\} = P_4 \cdot P\{4\,|\,4\} = P_4 \cdot P_4 = \frac{3}{36}\cdot\frac{3}{36} = \frac{9}{36^2}$

$$P\left\{4\ 1^{\text{st}},\ \text{lose } 2^{\text{nd}}\right\} = P\{4,7\} = P_4 \cdot P\{7\,|\,4\} = P_4 \cdot P_7 = \frac{3}{36}\cdot\frac{6}{36} = \frac{18}{36^2}$$

The last inequalities are due to the fact that subsequent rolls are independent experiments.

3) In order to win (lose) at the *n*-th roll, you have to obtain a result which is not in {4,7} for *n*-2 times before getting a 4 (7) again.

$$P\left\{4\ 1^{\text{st}},\ \text{win } n^{th}\right\} = P\left\{4,\left[\sim(4,7)^{n-2}\right],4\right\} = P_4\cdot\left(1-P_4-P_7\right)^{n-2}\cdot P_4 = \frac{3}{36}\cdot\left(\frac{27}{36}\right)^{n-2}\cdot\frac{3}{36}$$

$$P\left\{4\ 1^{\text{st}},\ \text{lose } n^{th}\right\} = P\left\{4,\left[\sim(4,7)^{n-2}\right],7\right\} = P_4\cdot\left(1-P_4-P_7\right)^{n-2}\cdot P_7 = \frac{3}{36}\cdot\left(\frac{27}{36}\right)^{n-2}\cdot\frac{6}{36}$$

4) We straightforwardly obtain:

$$\begin{aligned}
P_{win} &= P\left\{\text{win } 1^{\text{st}}\right\} + \sum_{n=2}^{+\infty}\sum_{x\in\{4,5,6,8,9,10\}} P_x^{\,2}\cdot\left(1-P_x-P_7\right)^{n-2} \\
&= P\left\{\text{win } 1^{\text{st}}\right\} + \sum_{x\in\{4,5,6,8,9,10\}} P_x^{\,2}\cdot\sum_{n=2}^{+\infty}\left(1-P_x-P_7\right)^{n-2} \\
&= P\left\{\text{win } 1^{\text{st}}\right\} + \sum_{x\in\{4,5,6,8,9,10\}} \frac{P_x^{\,2}}{P_x+P_7} \\
&= \frac{8}{36} + 2\cdot\left[\frac{\left(\frac{3}{36}\right)^2}{\left(\frac{3+6}{36}\right)} + \frac{\left(\frac{4}{36}\right)^2}{\left(\frac{4+6}{36}\right)} + \frac{\left(\frac{5}{36}\right)^2}{\left(\frac{5+6}{36}\right)}\right] \\
&= \frac{8}{36} + 2\cdot\left[\frac{\left(\frac{9}{36}\right)}{9} + \frac{\left(\frac{16}{36}\right)}{10} + \frac{\left(\frac{25}{36}\right)}{11}\right] \\
&= \frac{2}{9} + \frac{1}{18} + \frac{4}{45} + \frac{25}{198} \\
&= \frac{244}{495} \approx 0.493
\end{aligned}$$

5) The expected payoff is $E[X] = 100\cdot P_{win} - 100\cdot\left(1-P_{win}\right) = 100\cdot\left(2P_{win}-1\right) = -\frac{700}{495}$