Exercise 2

Given a Bernoulli RV X with $p = 1/3$, and a RV Y such that:

- P(Y=1|X=0)= 0.15,
- P(Y=2|X=0)= 0.45,
- P(Y=3|X=0)= 0.3,
- P(Y=1|X=1)= 0.6,
- P(Y=2|X=1)= 0.3,
- P(Y=3|X=1)= 0.3.

1) compute the JPMF of (X,Y) and the PMF of Y.
2) compute mean and variance of Y.
3) discuss whether X and Y are independent and/or identically distributed. Explain your findings.
4) compute the covariance between X and Y.

Consider now *n* RVs iid U_i distributed as $U_i \sim X/Y$.

5) determine the distribution of $U = \frac{1}{n}\sum_{i=1}^{n} U_i$ when $n = 1$ and when $n \to \infty$. Explain your findings.

Solution

1) The JPMF is shown in the table below, obtained from $P(X = j, Y = k) = P(Y = k \mid X = j) \cdot P(X = j)$

X Y	1	2	3
0	0.1	0.3	0.2
1	0.2	0.1	0.1

Y's PMF is obtained from the total probability problem, $P(Y = k) = \sum_j P(Y = k \mid X = j) \cdot P(X = j)$, and it is equal to:

P(Y=1)=0.3, P(Y=2)=0.4, P(Y=3)=0.3.

2) E(Y)=2 and $Var(Y) = E(Y^2) - E(Y)^2$ =4.6 – 4 = 0.6.

3) X and Y are not identically distributed (which is fairly obvious, since Y has three values whereas X has two). They are not independent either, since:

$$P(X = 0, Y = 1) = 0.1$$
$$\neq$$
$$P(X = 0) \cdot P(Y = 1) = \frac{2}{3} \cdot 0.3 = 0.2$$

4) $Cov(X,Y) = E[X \cdot Y] - E[X] \cdot E[Y] = 0.7 - \frac{1}{3} \cdot 2 = -0.0333$.

5) When n=1 we obtain P(U=0)=0.6, P(U=1/3)=0.1, P(U=1/2)=0.1, P(U=1)=0.2. From this we obtain E(U)=0.28333. By the Central Limit Theorem, when $n \to \infty$ RV U is distributed as a Dirac delta, around its mean value E(U).